

Polydimensional Supersymmetric Principles*

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Abstract

Systems of equations are invariant under *polydimensional transformations* which reshuffle the geometry such that what is a line or a plane is dependent upon the frame of reference. This leads us to propose an extension of Clifford calculus in which each geometric element (vector, bivector) has its own coordinate. A new classical action principle is proposed in which particles take paths which minimize the distance traveled plus area swept out by the spin. This leads to a solution of the 50 year old conundrum of ‘what is the correct Lagrangian’ in which to derive the Papapetrou equations of motion for spinning particles in curved space (including torsion).

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1. Introduction

Reformulating physical laws with a new mathematical language will not in itself lead to new principles. However, because Clifford algebra (Lounesto, 1997) encodes the structure of the underlying geometric space, we see possible bigger patterns emerge. Specifically in the description of a spinning particle the equations of motion are invariant under a non-dimensional preserving *polydimensional* transformation which rotate between vector momentum and bivector spin. This leads us to propose that the physical laws might be covariant under general automorphism transformations which reshuffle the geometry, a classical analogy to the quantum model of Crawford (1994).

The invariants of these transformations suggest that the spin motion contributes to the proper time. Hence a new action principle is proposed in which particles take paths which minimize the sum of the linear distance traveled combined with the bivector area swept out by the spin. In curved space, the velocity of the variation is not the variation of the velocity, leading to a new derivation of the Papapetrou (1951) equations (of a spinning particle) as autoparallels in the polydimensional space.

2. Relative Dimensionalism

Is the dimension of the geometric quantity (e.g. scalar, vector) absolutely unique to the associated physical quantity? Certainly mass is a 0D (zero-dimensional) scalar while momentum a 1D vector. In contrast, consider that while time is viewed as the 4th dimension in Minkowski space, special relativity was originally formulated with time treated as a scalar. Is there a right/wrong answer as to the geometric nature of time, or is it a function of the observer's frame of reference? We suggest that *dimension is relative* (Pezzaglia, 1998a), such that we can consider transformations which reshuffle the basis geometry (e.g. vector line replaced by bivector plane), yet leave sets of physical laws invariant. One application provides a new derivation of the enhancement of the mechanical mass by the amount of spin.

2.1. Review of Special Relativity

In electrodynamics one can unify a 3D vector force law with a 3D scalar work-energy law,

$$\dot{\mathcal{E}} = e \vec{\mathbf{E}} \bullet \vec{\mathbf{v}} , \quad (1)$$

$$\dot{\vec{\mathbf{P}}} = e \left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}} \right) , \quad (2)$$

into one single equation,

$$\dot{p}^\mu = \left(\frac{e}{m} \right) p_\nu F^{\mu\nu} , \quad (3)$$

using 4D vectors (and tensors). Certainly the adoption of the four-dimensional viewpoint has notational economy, and provides insight that the work-energy

theorem (1) is simply the fourth aspect of the vector force law (3). However, philosophically one can ask if the 4D viewpoint is any more correct than the 3D equations as they describe the same phenomena. Since special relativity was originally formulated without the concept of Minkowski spacetime, it is convenient, but apparently not necessary to adopt the paradigm shift from 3D to 4D. Hence we are being purposely dialectic in raising the question whether one can make an absolute statement about the dimensional nature of a physical quantity such as time. Can we state (measure) that time is a part of a four-vector (as opposed to a 3D scalar), or is this relative to whether one adopts a 3D or 4D world view, hence relative to the observer's dimensional frame of reference?

Usually the behavior of quantities under symmetry transformations is used to define the dimensional nature (e.g. does a set of three quantities transform like a vector under rotations). In classical physics the fundamental laws must be invariant under rotational displacements because it is postulated that the universe is *isotropic* (has no preferred direction). When one formulates laws with vectors (which are inherently coordinate system independent), isotropy is 'built in' without needing to separately impose the condition. Hence (Gibbs) vectors are a natural language to express classical (3D) physical laws because they naturally encode isotropy. Einstein further postulated the metaprinciple that motion was relative; that there is no absolute preferred rest frame to the universe. This coupled with the postulate that the speed of light is the same for all observers leads to the principle that the laws of physics must be invariant under Lorentz transformations (which connect inertial frames of reference). As a consequence, in a 3D perspective, what is pure scalar (e.g. time interval) to one observer is part scalar, part vector to another observer. Lorentz transformations, which are rotations in 4D spacetime that preserve the dimension of the geometry, in a 3D viewpoint NOT dimensional preserving.

In 3D space the length (magnitude) of a vector (e.g. electric field or momentum) is invariant under rotations. Under Lorentz transformations (4D rotations), the modulus of the four-vector is invariant,

$$\| \mathbf{p} \|^2 \equiv p_\mu p^\mu = \mathcal{E}^2/c^2 - \| \vec{\mathbf{P}} \|^2 . \quad (4)$$

Reinterpreted with a 3D viewpoint, the invariant quantity is the difference between the square of the scalar energy minus the magnitude of the 3D momentum vector. Neither the modulus of the 3D scalar energy, nor 3D vector momentum is independently invariant under these transformations. Further, in the 3D viewpoint it is as if the mass of the particle (e.g. in definition of momentum: $p = mv$) has been increased by its kinetic energy content,

$$m \equiv m_0 \sqrt{1 + \left(\frac{\| \vec{\mathbf{P}} \|}{m_0 c} \right)^2} . \quad (5)$$

2.2. Automorphism Invariance

Physicists usually first encounter Clifford algebras in quantum mechanics in the form of Pauli, Majorana and Dirac ‘spin’ matrices. The spin-space analogy to isotropy is that the physical formulation must be covariant under global rotations of the spin basis. An equivalent metaprinciple would be to require that the physics is invariant under a change of representation of the Dirac matrices.

It is possible however to avoid talking about the matrix representation entirely. The more general concept is an algebra automorphism, which is a transformation of the basis generators γ_μ of the algebra which preserves the Clifford structure,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} , \quad (6)$$

where $g_{\mu\nu}$ is the spacetime metric. For example, consider the following orthogonal transformation on any element Γ of the Clifford algebra,

$$\Gamma' = \mathcal{R} \Gamma \mathcal{R}^{-1} , \quad (7)$$

$$\mathcal{R}(\phi) \equiv \exp(\gamma_\mu \phi^\mu / 2) , \quad \mu = 1, 2, 3, 4. \quad (8)$$

Proposing local covariance of the Dirac equation under such automorphism transformations is one path to gauge theories of gravity (see Crawford, 1994).

If the elements γ_μ are interpreted geometrically as basis vectors, then (8) reshuffles geometry. For example, when $\phi^4 = \pi/2$, equation (7) causes the permutation,

$$\gamma_j \iff \gamma_4 \gamma_j , \quad j = 1, 2, 3, \quad (9)$$

$$\gamma_1 \gamma_2 \gamma_3 \iff \gamma_4 \gamma_1 \gamma_2 \gamma_3 , \quad (10)$$

which exchanges three of the vectors with their associated timelike bivectors. What is a 1D vector in one “reference frame” is hence a 2D plane in another. The transformation (8) thus “rotates” vectors into planes.

2.3. Polydimensional Formulation

Just as four-vectors allowed us to unify two equations into one, the language of Clifford algebra allows for further notational economy. Consider that a classically spinning charged particle obeys the torque equation of motion,

$$\dot{S}^{\mu\beta} = \left(\frac{e}{m}\right) (F^\mu{}_\nu S^{\nu\beta} - F^\beta{}_\nu S^{\nu\mu}) . \quad (11)$$

This and (3) can be written in the single statement,

$$\dot{\mathcal{M}} = \left(\frac{e}{2m}\right) [\mathcal{M}, \mathbf{F}] , \quad (12)$$

where $\mathbf{F} = \frac{1}{2} F^{\mu\nu} \mathbf{e}_\mu \wedge \mathbf{e}_\nu$ is the electromagnetic field bivector and \mathbf{e}_μ are the basis vectors of the geometric space. The *momentum polyvector* (Pezzaglia, 1998b) is

defined as the multivector sum of the vector linear momentum and the bivector spin momentum,

$$\mathcal{M} \equiv p^\mu \mathbf{e}_\mu + \frac{1}{2\lambda} S^{\mu\nu} \mathbf{e}_\mu \wedge \mathbf{e}_\nu , \quad (13)$$

where λ is some fundamental length scale constant (to be interpreted in the next section). The ability to add different ranked (dimensional) geometries is the notational advantage of Clifford geometric algebra over standard tensors. Mathematically, (12) allows one to simultaneously obtain solutions to both equations (3) and (11).

It is interesting to note that (12) is invariant under the automorphism transformations generated by (8). For example, example, $\phi^4 = \pi/2$ in (9) causes a trading between momentum and mass moment of the spin tensor,

$$\lambda p_j \iff S_{4j} . \quad (14)$$

It is not at all clear what physical interpretation to ascribe to the two frames of reference. A radical assertion of the *principle of relative dimensionalism* (Pezzaglia, 1998a) would be to propose that what is a vector to one observer is a bivector to another, and that they would partition the polymomentum (13) into momentum and spin portions differently. What is spin to one would be momentum to the other.

Under the general rotation of the vectors into bivectors, both observers would agree that the following generalized modulus of the polyvector (13) would be invariant,

$$\| \mathcal{M} \|^2 \equiv p_\mu p^\mu + \lambda^{-2} S_{\mu\nu} S^{\nu\mu} . \quad (15)$$

In the $(- - +)$ metric signature we define the modulus to be the *bare mass*: $m_0 \equiv c^{-1} \| \mathcal{M} \|$. This implies that the mechanical mass (modulus of the linear momentum) is NOT invariant under these transformations, but has been enhanced by the spin energy content,

$$m \equiv c^{-1} \| \mathbf{p} \| = m_0 \sqrt{1 + \frac{S^{\mu\nu} S_{\mu\nu}}{(m_0 c \lambda)^2}} , \quad (16)$$

in analogy to (5). What we have described in (15), by simple geometric construction, is a familiar result, laboriously obtained by Dixon (1970) in the mechanical analysis of spinning bodies. Expanding (16) non-relativistically one sees that λ is consistent with the *radius of gyration* of a classical extended particle.

3. New Action Principle

The polymomenta gives the (vector) linear momenta and (bivector) spin momenta equal importance. We now propose that each quantity democratically has its own conjugate coordinate. The generalized action principle is that particles take the paths which minimize the sum of the linear distance traveled combined with the bivector area swept out. This simple geometric idea gives a new derivation of the spin enhanced mass described by the Dixon equation (15) and the Weysenhoff condition for spinning particles.

3.1. Review of Classical Mechanics

Classical particles will follow paths of least spacetime distance between end-points, even when the space is curved by gravity. The measure of distance between two points in flat spacetime is,

$$c^2 d\tau^2 \equiv c^2 dt^2 - (dx^2 + dy^2 + dz^2) = dx^\alpha dx^\beta g_{\alpha\beta} , \quad (17)$$

where affine parameter τ is commonly called the *proper time*. If we adopt the 3D viewpoint, we are combining (in quadrature) the ‘scalar’ time displacement with the ‘vector’ path displacement, utilizing a fundamental constant c (the speed of light) to combine the unlike quantities.

To obtain the equations of motion, one minimizes (extremizes) the *action integral*, which is based upon the quadratic form (17), [note $x^4 \equiv ct$],

$$\mathcal{A} \equiv \int \mathcal{L} d\tau = \int m_0 c d\tau = \int m_0 c \sqrt{\dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta}} d\tau . \quad (18)$$

The integrand \mathcal{L} is called the *Lagrangian*, which is generally a function of the coordinates x^α , and the velocities: $\dot{x}^\alpha = dx^\alpha/d\tau$ relative to the proper time.

The canonical *four-momentum* p_μ is defined,

$$p_\mu \equiv \frac{\delta \mathcal{L}}{\delta u^\mu} = m_0 u_\mu = m_0 \dot{x}_\mu , \quad (19)$$

which obeys (4). It is easy to show that the 3D part of the momentum $P_j = mv_j$ has mass m which is enhanced by the energy content according to (5).

3.2. Dimensional Democracy

If we fully embrace the concept of *relative dimensionalism*, then we must recognize that what one observer labels as a ‘point’ in spacetime with vector coordinates (t, x, y, z) may be seen as an entirely different geometric object by another. This suggests that perhaps we should formulate physics in a way which is completely *dimensionally democratic* (Pezzaglia, 1998b) in that all ranks of geometry are equally represented. We propose therefore that ‘the world’ is not the usual four-dimensional manifold, but instead a fully *polydimensional continuum*, made of points, lines, planes, etc. Each event Σ is a geometric point in a *Clifford manifold* (Chisholm and Farwell, 1991), which has a coordinate q^A associated with each basis element \mathbf{E}_A (vector, bivector, trivector, etc.). The *pandimensional differential* in the manifold would be,

$$d\Sigma \equiv \mathbf{E}_A dq^A = \mathbf{e}_\mu dx^\mu + \frac{1}{2\lambda} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta da^{\alpha\beta} + \frac{1}{6\lambda^2} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\sigma dV^{\alpha\beta\sigma} + \dots , \quad (20)$$

where in Clifford algebra it is perfectly valid to add vectors to planes and volumes (parameterized by the antisymmetric tensor coordinates $dx^\mu, da^{\alpha\beta}, dV^{\alpha\beta\sigma}$ respectively).

In analogy to (15), we propose that the quadratic form of the Clifford manifold would be the scalar part of the square of (20)

$$\| d\Sigma \|^2 \equiv dx^\mu dx_\mu + \frac{1}{2\lambda^2} da^{\alpha\beta} da_{\beta\alpha} + \frac{1}{6\lambda^4} dV^{\alpha\beta\sigma} dV_{\sigma\beta\alpha} + \dots \quad (21)$$

The fundamental length constant λ must be introduced in order to add the bivector ‘area’ coordinate contribution to the vector ‘linear’ one (Pezzaglia, 1998b). This suggests that we have a new affine parameter $d\kappa = \| d\Sigma \|$ which we will use to parameterize our ‘polydimensional’ equations of motion.

Classical mechanics assumes point particles that trace out linear paths. The equations of motion are based upon minimizing the distance of the path. String theory introduces one-dimensional objects which trace out areas, and the equations of motion are analogously based upon minimizing the total area. Membrane theory proposes two-dimensional objects which trace out (three-dimensional) volumes to be minimized. Our new action principle suggests that we should add all of these contributions together, and treat particles as polygeometric objects which trace out polydimensional paths with (21) the quantity to be minimized.

3.3. Application to the Classical Spinning Particle

Using only the vector and bivector contributions of (21) the Lagrangian that is analogous to (18) would be,

$$\mathcal{L}(x^\alpha, \overset{\circ}{x}^\alpha, a^{\alpha\beta}, \overset{\circ}{a}^{\alpha\beta}) = m_0 c \sqrt{\overset{\circ}{x}^\mu \overset{\circ}{x}^\nu g_{\mu\nu} + \frac{1}{2\lambda^2} \overset{\circ}{a}^{\alpha\beta} \overset{\circ}{a}^{\mu\nu} g_{\beta\mu} g_{\alpha\nu}}, \quad (22)$$

where the open dot denotes differentiation with respect to the new affine parameter (whereas the small dot is with respect to the proper time),

$$\overset{\circ}{Q} \equiv \frac{dQ}{d\kappa} = \dot{Q} \frac{d\tau}{d\kappa}. \quad (23)$$

The relationship of the new affine parameter $d\kappa$ to the proper time $d\tau$ is easily derived by dividing (21) by $d\tau$ or $d\kappa$, noting $d\tau^2 = dx^\mu dx_\mu$,

$$\frac{d\tau}{d\kappa} \equiv \left(1 - \frac{\dot{a}^{\mu\nu} \dot{a}_{\mu\nu}}{2c^2 \lambda^2} \right)^{-1/2} = \sqrt{1 + \frac{\overset{\circ}{a}^{\mu\nu} \overset{\circ}{a}_{\mu\nu}}{2c^2 \lambda^2}}. \quad (24)$$

We interpret the spin to be the canonical momenta conjugate to the bivector coordinate,

$$S_{\mu\nu} \equiv \lambda^2 \frac{\delta \mathcal{L}}{\delta \overset{\circ}{a}^{\mu\nu}} = m_0 \overset{\circ}{a}_{\mu\nu} = m \dot{a}_{\mu\nu}, \quad (25)$$

$$p_\mu \equiv \frac{\delta \mathcal{L}}{\delta \overset{\circ}{x}^\mu} = m_0 \overset{\circ}{x}_\mu = m \dot{x}_\mu. \quad (26)$$

These definitions of the momenta satisfy the Dixon equation (15). When these momenta are reparameterized in terms of the more familiar proper time, they have spin enhanced mass: $m = m_0 d\tau/d\kappa$, consistent with (16).

Its easy to see that the Lagrangian (22) is invariant under the polydimensional coordinate rotation (between vectors and bivectors), generated by the four arbitrary parameters $\delta\phi^\alpha$ of the automorphism transformation (8),

$$\delta x^\alpha = \lambda^{-1} \delta\phi^\mu a_\mu^\alpha, \quad (27)$$

$$\delta a^{\mu\nu} = \delta\phi^\mu x^\nu - \delta\phi^\nu x^\mu. \quad (28)$$

Noether's theorem associates with this symmetry transformation a new set of constants of motion,

$$Q_\mu = \frac{\delta\mathcal{L}}{\delta\dot{x}^\alpha} \frac{\delta x^\alpha}{\delta\phi^\mu} + \frac{1}{2} \frac{\delta\mathcal{L}}{\delta\dot{a}^{\alpha\beta}} \frac{\delta a^{\alpha\beta}}{\delta\phi^\mu} = a_\mu^\alpha p_\alpha + S_{\mu\beta} x^\beta. \quad (29)$$

Taking the derivative of (29) with respect to the affine parameter yields the familiar Weysenhoff condition,

$$p_\mu S^{\mu\nu} = 0. \quad (30)$$

This is quite significant, because usually (30) is imposed at the onset by fiat, while we have provided an actual derivation based on the new automorphism symmetry of the Lagrangian!

4. General Poly-Covariance

In general we find that particles will deviate from geodesics due to contributions from derivatives of the basis vectors with respect to the new bivector coordinate. Further, in classical mechanics the variation of the velocity is no longer equal to the velocity of the variation. This leads to a new derivation of the Papapetrou equations (Papapetrou, 1951) describing the motion of spinning particles in curved space.

4.1. Covariant Derivatives in the Clifford Manifold

The total derivative of a basis vector with respect to the new affine parameter (24) must by the chain rule contain a derivative with respect to the bivector coordinate,

$$\dot{\mathbf{e}}_\mu \equiv \frac{d\mathbf{e}_\mu}{d\kappa} = \dot{x}^\sigma \frac{\partial \mathbf{e}_\mu}{\partial x^\sigma} + \frac{1}{2} \dot{a}^{\alpha\beta} \frac{\partial \mathbf{e}_\mu}{\partial a^{\alpha\beta}}. \quad (31)$$

Our ansätze is (Pezzaglia, 1999) that the bivector derivative obeys,

$$\frac{\partial \mathbf{e}_\mu}{\partial a^{\alpha\beta}} = [\partial_\alpha, \partial_\beta] \mathbf{e}_\mu - \tau_{\alpha\beta}^\sigma \partial_\sigma \mathbf{e}_\mu = (R_{\alpha\beta\mu}{}^\nu - \tau_{\alpha\beta}^\sigma \Gamma_{\sigma\mu}^\nu) \mathbf{e}_\nu, \quad (32)$$

where $\tau_{\alpha\beta}^\sigma$ is the torsion, $\Gamma_{\sigma\mu}^\nu$ the Cartan connection and $R_{\alpha\beta\mu}{}^\nu$ the Cartan curvature.

We can factor out the basis vectors by defining the covariant derivative,

$$\frac{\partial}{\partial x^\mu} (p^\nu \mathbf{e}_\nu) = \mathbf{e}_\nu \nabla_\mu p^\nu \equiv \mathbf{e}_\nu (\partial_\mu p^\nu + p^\sigma \Gamma_{\mu\sigma}^\nu) , \quad (33)$$

$$\frac{\partial}{\partial a^{\alpha\beta}} (p^\nu \mathbf{e}_\nu) = \mathbf{e}_\nu [\nabla_\alpha, \nabla_\beta] p^\nu \equiv \mathbf{e}_\nu (R_{\alpha\beta\mu}^\nu p^\mu - \tau_{\alpha\beta}^\sigma \nabla_\sigma p^\nu) . \quad (34)$$

From these definitions it is clear than the covariant derivatives of the basis vectors vanish as usual.

The parallel transport of the conserved canonical momenta generates autoparallels in the Clifford manifold,

$$0 = \frac{d}{d\kappa} (\mathbf{e}_\mu p^\mu) = \mathbf{e}_\mu \left(\dot{x}^\sigma \nabla_\sigma + \frac{1}{2} \dot{a}^{\alpha\beta} [\nabla_\alpha, \nabla_\beta] \right) p^\mu , \quad (35)$$

$$0 = \frac{d}{d\kappa} (\mathbf{e}_{\mu\nu} S^{\mu\nu}) = \mathbf{e}_{\mu\nu} \left(\dot{x}^\sigma \nabla_\sigma + \frac{1}{2} \dot{a}^{\alpha\beta} [\nabla_\alpha, \nabla_\beta] \right) S^{\mu\nu} , \quad (36)$$

where $\mathbf{e}_{\mu\nu} \equiv \mathbf{e}_\mu \wedge \mathbf{e}_\nu$. Substituting (33) and (34), the above equations provide a new derivation of the Papapetrou equations of motion for spinning particles (Papapetrou, 1951). Ours however are more general as they include torsion and all the higher order terms. In contravariant form,

$$0 = \dot{p}^\mu + \left(\dot{x}^\sigma \Gamma_{\sigma\nu}^\mu + \frac{1}{2} \dot{a}^{\alpha\beta} R'_{\alpha\beta\nu}{}^\mu \right) p^\nu , \quad (37)$$

$$0 = \dot{S}^{\rho\omega} + \delta_{\lambda\sigma}^{\rho\omega} \left(\dot{x}^\alpha \Gamma_{\alpha\nu}^\lambda + \frac{1}{2} \dot{a}^{\alpha\beta} R'_{\alpha\beta\nu}{}^\lambda \right) S^{\nu\sigma} , \quad (38)$$

$$R'_{\alpha\beta\nu}{}^\mu \equiv R_{\alpha\beta\nu}{}^\mu - \tau_{\alpha\beta}^\sigma \Gamma_{\sigma\nu}^\mu . \quad (39)$$

4.2. An-Holonomic Mechanics

It has been a long-standing unsolved problem to derive the Papapetrou equations from a simple Lagrangian. We succeed where so many others have failed because of our definition of the new affine parameter, the form of the Lagrangian (22) and by noting that the introduction of the bivector coordinate has made the system an-holonomic. Consider the variation of the Lagrangian,

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta x^\alpha} \delta x^\alpha + \frac{\delta \mathcal{L}}{\delta \dot{x}^\alpha} \delta \dot{x}^\alpha + \frac{1}{2} \frac{\delta \mathcal{L}}{\delta a^{\alpha\beta}} \delta a^{\alpha\beta} + \frac{1}{2} \frac{\delta \mathcal{L}}{\delta \dot{a}^{\alpha\beta}} \delta \dot{a}^{\alpha\beta} . \quad (40)$$

To get the equations of motion, the terms proportional to variations of velocities must be rewritten in terms of variations of the coordinates. Integrating the second term on the right by parts,

$$\frac{\delta \mathcal{L}}{\delta \dot{x}^\alpha} \delta \dot{x}^\alpha = p_\alpha \delta \dot{x}^\alpha = \frac{d}{d\kappa} (p_\alpha \delta x^\alpha) - \dot{p}_\alpha \delta x^\alpha + p_\alpha \left(\delta \dot{x}^\alpha - \frac{d}{d\kappa} \delta x^\alpha \right) . \quad (41)$$

The leading term on the right does not contribute to the equations of motion (the path is varied with fixed endpoints).

It is usually assumed in most undergraduate texts that the velocity of the variation is equal to the variation of the velocity such that the last term of (41) vanishes. This is no longer necessarily true when the coordinate system is anholonomic as is our case with bivector coordinates and path dependent basis vectors. We assert that in general the following is valid,

$$\delta \left(\overset{\circ}{x}^\mu \mathbf{e}_\mu \right) = \frac{d}{d\kappa} (\delta x^\mu \mathbf{e}_\mu) , \quad (42)$$

$$\delta \left(\overset{\circ}{a}^{\alpha\beta} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \right) = \frac{d}{d\kappa} (\delta a^{\alpha\beta} \mathbf{e}_\alpha \wedge \mathbf{e}_\beta) . \quad (43)$$

A lengthy proof involving anholonomic coordinate transformations will appear in Pezzaglia (1999). In principle the derivation is an extension of the method introduced by Kleinert (1997) for spaces with torsion. Performing the variations and derivatives in the above equations and rearranging terms gives us,

$$\left(\delta \overset{\circ}{x}^\mu - \frac{d\delta x^\mu}{d\kappa} \right) = \delta x^\alpha \overset{\circ}{x}^\beta \tau_{\alpha\beta}^\sigma + \frac{1}{2} \left(\delta x^\alpha \overset{\circ}{a}^{\mu\nu} - \overset{\circ}{x}^\alpha \delta a^{\mu\nu} \right) R'_{\mu\nu\alpha}{}^\sigma , \quad (44)$$

$$\left(\delta \overset{\circ}{a}^{\mu\nu} - \frac{d\delta a^{\mu\nu}}{d\kappa} \right) = \delta_{\omega\nu}^{\lambda\sigma} \left[\Gamma_{\alpha\mu}^\omega \left(\overset{\circ}{x}^\alpha \delta a^{\mu\nu} - \overset{\circ}{a}^{\mu\nu} \delta x^\alpha \right) + \frac{1}{4} R'_{\alpha\beta\mu}{}^\omega \left(\overset{\circ}{a}^{\mu\nu} \delta a^{\alpha\beta} - \overset{\circ}{a}^{\alpha\beta} \delta a^{\mu\nu} \right) \right] . \quad (45)$$

The first term on the right in (44) involving the torsion follows Kleinert (1997), the rest are new. Substituting (44) into (41) and back into (40) and doing the same for the (45), collecting terms proportional to δx^μ , we obtain the anholonomic form of the Euler-Lagrange equations of motion,

$$\frac{\delta \mathcal{L}}{\delta x^\mu} - \overset{\circ}{p}_\mu + p_\lambda \overset{\circ}{x}^\alpha \tau_{\alpha\mu}^\lambda + \left(\frac{1}{2} p_\lambda R'_{\alpha\beta\mu}{}^\lambda + S_{\omega\beta} \Gamma_{\mu\alpha}^\omega \right) \overset{\circ}{a}^{\alpha\beta} = 0 , \quad (46)$$

where R' is defined (39). The first two terms are the standard, the third appears in Kleinert (1997), the rest are new. Performing the derivative on the Lagrangian (22) we recover the covariant form of the Papapetrou equations (37). A parallel construction will yield the spin equation (38).

4.3. Metamorphic Covariance

Our Lagrangian (22) is invariant under *local* automorphism transformations, where in general the ϕ^μ of (8), (27) and (28) can be position dependent upon a path integral of a gauge field,

$$\phi^\nu(x^\alpha) = \int^{x^\alpha} B^\nu{}_\mu(y^\sigma) dy^\mu . \quad (47)$$

This would imply that the connection of a basis vector would become *geometa-morphic* (Pezzaglia, 1998a), e.g. under parallel transport a vector will turn a plane,

$$\partial_\sigma \mathbf{e}_\nu = \mathbf{e}_\alpha \Gamma_{\sigma\nu}^\alpha + \mathbf{e}_\mu \wedge \mathbf{e}_\nu B^\mu{}_\sigma . \quad (48)$$

Obviously this would have impact on equations (31) through (46) of this paper.

Equation (48) is the classical analog to Crawford's (1994) spin covariant covariant derivative for the Dirac equation derived from generalized automorphism transformations of the Dirac algebra,

$$\nabla_\mu = \partial_\mu + i(eA_\mu + \gamma^5 a_\mu) + \gamma_\nu \left(\frac{1}{2} B^\nu_\mu + \gamma^5 i b^\nu_\mu \right) + \frac{1}{2} \gamma_{\alpha\beta} C^{\alpha\beta}_\mu, \quad (49)$$

$$(-i\hbar\gamma^\mu\nabla_\mu - mc)\psi = 0. \quad (50)$$

The gauge field B^μ_σ is the same in (48) and (49).

The Dirac equation is obtained more or less by factoring (4) into a linear operator and replacing the momentum by the gauge-covariant derivative: $p_\mu \rightarrow -i\hbar\nabla_\mu$. We propose that a generalized equation might be derived from factoring the Dixon equation (15), and associating the commutator derivative with the spin operator $S_{\mu\nu} \rightarrow -i\hbar\lambda^2 [\nabla_\mu, \nabla_\nu]$. Thus we postulate the form,

$$\left(-i\hbar\gamma^\mu\nabla_\mu - i\hbar\frac{\lambda}{2}\gamma^{\alpha\beta}[\nabla_\alpha, \nabla_\beta] - m_0c \right) \psi = 0. \quad (51)$$

Certainly one could include higher order triple commutator derivatives. In flat space with all but the electromagnetic gauge field A_μ suppressed in (49), the bivector (commutator) derivative will introduce an anomalous magnetic moment interaction which provides a possible interpretation of the constant λ .

5. Summary

In introducing *Dimensional Democracy* we have given the bivector a coordinate and show its utility in the treatment of the classical spinning particle problem. This system is invariant under *polydimensional* transformations which reshuffle geometry such that 'what is a vector' is *dimensionally relative* to the observer's frame. A fundamentally new action principle has been introduced which can accomodate anholonomic systems with torsion and spin. Most important, the principles proposed have potential broad applications beyond the examples in this paper.

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